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On Piecewise Polynomial Interpolation in Rectangular Polygons

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1. INTRODUCTION

This paper deals with the technical problem of developing well-set schemes for *interpolation* by bivariate *spline* functions in subdivided rectangular polygons.

In Section 3, we derive a family of interpolation schemes for polynomial splines of arbitrary degree and smoothness. These schemes are *algebraically* well-set in the sense [2, p. 169] that, for any smooth function (or set of values and derivatives) f and any mesh π, a *unique* spline interpolant exists having the specified values and derivatives.

A variational characterization of bicubic splines over rectangular polygons is discussed in [1, p. 254–255]. In Section 4, evidence is presented which shows this is but a *partial* generalization of the corresponding result for rectangles [1, Theorem 7.6.1].

In Section 5, we discuss the more difficult question of finding bivariate spline interpolation schemes which are *analytically well-set* in the sense that, as the mesh π is successively refined, the associated sequence of interpolants of any sufficiently smooth function f converges to f. It is noted that often the interpolation schemes of Section 3 are not analytically well set. We then give an alternate, analytically well-set interpolation scheme for bicubic splines over L-shaped regions, thus answering affirmatively a query of Birkhoff and de Boor [2, Appendix A, p. 187].

2. NOTATION AND PRELIMINARY RESULTS

We recall some definitions and notation from [5] and [6]. Let $\mathscr{R}$ be any rectangular polygon, and let π be a rectangular mesh containing every corner

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of $\mathscr{R}$ as a mesh point. Let $P^n(\mathscr{R}, \pi)$ denote the linear space of piecewise polynomial functions of degree $2n - 1$ in each variable. Then $H^n(\mathscr{R}, \pi) \equiv P^n(\mathscr{R}, \pi) \cap C^{n-1}(\mathscr{R})$ is called the *smooth Hermite space of order n*.

We define for each n the chain of subspaces

$$S_k^n(\mathscr{R}, \pi) \equiv H^n(\mathscr{R}, \pi) \cap C^{n-1+k}(\mathscr{R}), \qquad 0 \leqslant k \leqslant n-1.$$

In the terminology of [1], $S_k^n(\mathscr{R}, \pi)$ is a *spline* subspace of *deficiency* $n - k$.[1]

Let $\mathscr{R}$ be a *rectangle* and consider the space $S_1^2(\mathscr{R}, \pi)$ of bicubic splines. In [7], de Boor showed that for a given (sufficiently differentiable) function f defined on $\mathscr{R}$, there exists a unique bicubic spline $s_f(x, y)$ such that (i) s_f interpolates to f at each mesh point of π, (ii) the normal derivative of s_f interpolates to the normal derivative of f at boundary mesh points, and (iii) the second order cross derivative of s_f interpolates to f_{xy} at the four corners of $\mathscr{R}$.

In this paper we extend de Boor's results by determining bases of interpolating conditions for each space $S_k^n(\mathscr{R}, \pi)$ over a general rectangular polygon $\mathscr{R}$. That is, we derive algebraically well-set interpolation schemes for each space $S_k^n(\mathscr{R}, \pi)$. These extensions of de Boor's scheme also depend upon various cross-derivatives being specified at *four* corner points. In most cases, these four corner points are neither *unique* nor *arbitrary*. Thus we must specify a set of conditions by which it can be determined whether a given set S of four corner points is suitable. The essential condition is that S *span* π in the sense that successive augmentation of S by mesh points on mesh lines in $(\mathscr{R}, \pi)$ through pairs of mesh points already included in S, ultimately gives all mesh points of $(\mathscr{R}, \pi)$. This condition asserts that for $S_0 = S$, $S_r = S_{r-1} \cup \{P_{ij} \in \pi : P_{ij}$ lies on some mesh line passing through two distinct points in $S_{r-1}\}$, $r = 1, 2, \ldots$, some S_N contains all points of π. Thus we define

DEFINITION 1. A set $S = \{Q_s = (x_s, y_s) : 1 \leqslant s \leqslant 4\}$ of four corner points is an *amenable set* if and only if (i) some pair but no triple of the points in S lie on the same mesh line, and (ii) S *spans* π.

To illustrate amenable and nonamenable sets of corner points, consider the polygon shown in Fig. 1. The set $\{Q_1, Q_2, Q_3, Q_4\}$ is an amenable set with $N = 4$, whereas $\{Q_1, Q_2, Q_3, Q_5)$ is *not* an amenable set. It can be shown that every rectangular polygon contains an amenable set of boundary points.

The proof of Theorems 1 and 3 are dependent on the following well-known univariate result [11, p. 122]:

[1] Similarly, $S_k^n(I, \pi) \equiv H^n(I, \pi) \cap C^{n-1+k}(I)$ is the spline subspace of the *univariate* smooth Hermite space $H^n(I, \pi)$, of deficiency $n - k$, where $I = [a, b]$ and $\pi : a = x_0 < \cdots < x_m = b$.

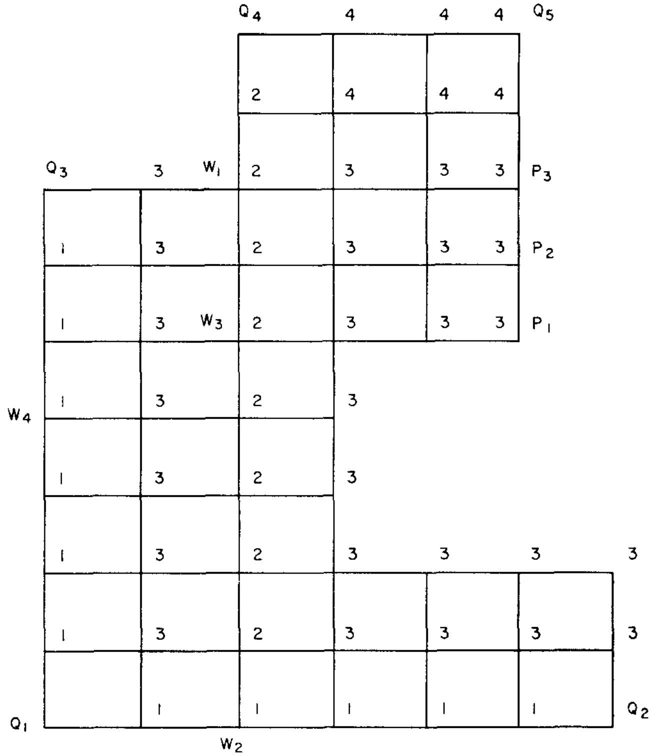


FIGURE 1

LEMMA 1. *Given the interval $I = [a, b]$ with partition $\pi : a = x_0 < \cdots < x_m = b$, the set of values $\{f_i^{(r)} : 0 \leqslant i \leqslant m, 0 \leqslant r \leqslant n-1\}$ and fixed indices $0 \leqslant \alpha < \beta \leqslant m$, there exists (for each k, $0 \leqslant k \leqslant n-1$) a unique $p_k \in S_k{}^n(I, \pi)$ such that*

$$p_k^{(r)}(x_i) = f_i^{(r)}, \tag{1}$$

where $0 \leqslant r \leqslant n-1$ for $i = \alpha, \beta$ and $0 \leqslant r \leqslant n-1-k$ for $i \neq \alpha, \beta$. [Here and below $f_i^{(r)} \equiv d^r f/dx^r(x_i)$.]

COROLLARY. *The dimension of $S_k{}^n(I, \pi)$ is $n(m+1) - k(m-1)$.*

Proof. For k *fixed*, define the $(m-1)(n-k) + 2n$ functions $\phi_{ir}(x)$ to be the unique elements (which exist by Lemma 1) of $S_k{}^n(I, \pi)$ such that for $0 \leqslant i \leqslant m$,

$$\phi_{ir}^{(s)}(x_j) = \delta_{ij}\delta_{rs} \begin{cases} i \neq \alpha, \beta; & 0 \leqslant j \leqslant m; \quad 0 \leqslant r, s \leqslant n-1-k, \\ i = \alpha, \beta; & 0 \leqslant j \leqslant m; \quad 0 \leqslant r, s \leqslant n-1. \end{cases} \tag{2}$$

Clearly these functions form an interpolation basis for $S_k{}^n(I, \pi)$, i.e., if

$$p_k(x) \equiv \sum_{r=0}^{n-1} \{f_\alpha^{(r)} \phi_{\alpha r}(x) + f_\beta^{(r)} \phi_{\beta r}(x)\} + \sum_{\substack{i=0 \\ i \neq \alpha,\beta}}^{m} \sum_{r=0}^{n-1-k} f_i^{(r)} \phi_{ir}(x), \tag{3}$$

then p_k satisfies (1) and the proof is complete.

For a *rectangle* $\mathscr{R} = I_1 \times I_2$, $S_k{}^n(\mathscr{R}, \pi) = S_k{}^n(I_1, \pi_1) \otimes S_k{}^n(I_2, \pi_2)$. Therefore, a basis for $S_k{}^n(\mathscr{R}, \pi)$ can be constructed as a tensor product of bases for $S_k{}^n(I_1, \pi_1)$ and $S_k{}^n(I_2, \pi_2)$ [9, p. 40]. Thus, if $p \in S_k{}^n(\mathscr{R}, \pi)$,

$$p(x, y) = \sum_i \sum_j \sum_r \sum_s p_k^{(r,s)}(x_i, y_j)\, \phi_{ir}(x)\, \psi_{js}(y), \tag{4}$$

where $\{\phi_{ir}(x)\}$ is the basis for $S_k{}^n(I_1, \pi_1)$ given in (2) and $\{\psi_{js}(y)\}$ is the corresponding basis for $S_k{}^n(I_2, \pi_2)$. The summation is over those indices for which the ϕ_{ir} and ψ_{js} are defined. It follows that $p \in C^{(n-1+k, n-1+k)}[\mathscr{R}]$, where $C^{(r,s)}[\mathscr{R}] \equiv \{f : f^{(i,j)} \equiv \partial^{(i+j)} f / \partial x^i \, \partial y^j$ is continuous in $\mathscr{R}$ for $0 \leqslant i \leqslant r, 0 \leqslant j \leqslant s\}$.

Lemma 1, therefore, has the following bivariate analog:

THEOREM 1. *Let* $(\mathscr{R}, \pi)$ *be a partitioned rectangle and let the values* $\{f_{ij}^{(r,s)} : 0 \leqslant r, s \leqslant n - 1\}$ *be given at each mesh point* $(x_i, y_j) \in \pi$. *For fixed indices* $0 \leqslant \alpha_1 < \beta_1 \leqslant m$, $0 \leqslant \alpha_2 < \beta_2 \leqslant m'$ *and for each* k, $0 \leqslant k \leqslant n - 1$, *there exists a unique* $p_k(x, y) \in S_k{}^n(\mathscr{R}, \pi)$ *such that*

$$p_k^{(r,s)}(x_i, y_j) = f_{ij}^{(r,s)}, \tag{5}$$

where the admissible values of the indices (r, s) *are given in Table I.*

TABLE I

Range of Indices (r, s) for Equation (5)

$r \backslash s$	$0\ 1 \cdots (n-1-k)$	$(n-k) \cdots (n-1)$
0, 1, $\vdots$, $(n-1-k)$	all mesh points (x_i, y_j)	$j = \alpha_2, \beta_2$
$(n-k)$, $\vdots$, $(n-1)$	$i = \alpha_1, \beta_1$	$i = \alpha_1, \beta_1$; $j = \alpha_2, \beta_2$

Proof. Set $p_k^{(r,s)}(x_i, y_j) = f_{ij}^{(r,s)}$ in (4) for the values of (r, s) and (i, j) specified in Table I. The result follows from the uniqueness of the representation in (4).

Remark. The classic interpolation problem for a rectangle $\mathscr{R}$ involves $\alpha_1 = \alpha_2 = 0$, $\beta_1 = m$ and $\beta_2 = m'$. Thus, the interpolation scheme of de Boor [7, Theorem 2] is included in Theorem 1. Theorem 1 implies that the role of the boundary mesh lines can be interchanged with those of interior mesh lines and the resulting scheme is still algebraically well-set.

The tensor product formulation of $S_k^n(\mathscr{R}, \pi)$ (where $\mathscr{R}$ is a rectangle) enables one to derive many different algebraically well-set interpolation schemes quite easily, as illustrated by Theorem 1. In contrast, for a general rectangular polygon $\mathscr{R}$, $S_k^n(\mathscr{R}, \pi)$ is not a tensor product of spaces of univariate splines (cf. Example after Theorem 4). Therefore, the development of an algebraically well-set scheme for the rectangular polygon is considerably more difficult. A result needed in this development and which is interesting in its own right is

THEOREM 2. *If $(\mathscr{R}, \pi)$ is a partitioned rectangular polygon, then for each k, $0 \leqslant k \leqslant n - 1$, $p_k \in S_k^n(\mathscr{R}, \pi)$ implies that $p_k \in C^{(n-1+k, n-1+k)}[\mathscr{R}]$.*

Proof. In the interior of each rectangular element $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ of $(\mathscr{R}, \pi)$, $p_k(x, y)$ is a polynomial. Thus, if the conclusion fails, it must do so at some point P common to two rectangular elements $\mathbf{R}_1$ and $\mathbf{R}_2$ (Fig. 2). Consider the point P' as the origin, and let p_k be given in $\mathbf{R}_l$ by

$$p_k(x, y) = \sum_{j=0}^{2n-1} \sum_{i=0}^{2n-1} a_{ij}^{(l)} x^i y^j, \qquad (l = 1, 2).$$

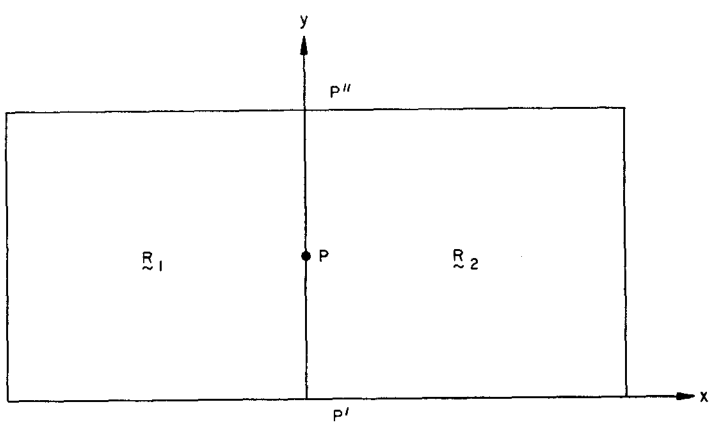


FIGURE 2

Since $p_k \in C[\mathscr{R}]$, the polynomial $p_k(0, y)$ is continuous along $\overline{P'P''}$ and thus $a_{0j}^{(1)} = a_{0j}^{(2)}$ for $0 \leqslant j \leqslant 2n - 1$. Similarly, $p_k^{(r,0)} \in C[\mathscr{R}]$ for each $0 \leqslant r \leqslant n - 1 + k$ implies $a_{rj}^{(1)} = a_{rj}^{(2)}$ for $0 \leqslant j \leqslant 2n - 1$. Hence $p_k^{(r,s)}$, $0 \leqslant r$, $s \leqslant n - 1 + k$, are continuous along $\overline{P'P''}$; the existence of the point P is contradicted, and the proof is complete.

3. Interpolation Schemes for Rectangular Polygons

We now establish our main result:

THEOREM 3. *Let $(\mathscr{R}, \pi)$ be a partitioned rectangular polygon. Let the values $f_{ij}^{(r,s)} : 0 \leqslant r,\ s \leqslant n - 1$ be given at each mesh point $(x_i, y_j) \in \pi$. Let S be a fixed amenable set of corner points. Then for each k, $0 \leqslant k \leqslant n - 1$, there exists a unique $p_k \in S_k{}^n(\mathscr{R}, \pi)$ such that*

$$p_k^{(r,s)}(x_i, y_j) = f_{ij}^{(r,s)}, \tag{6}$$

where the admissible values of the indices (r, s) are given in Table II.

TABLE II

Range of Indices (r, s) for Equation (6)

$r \diagdown s$	$0\ 1 \cdots (n-1-k)$	$(n-k) \cdots (n-1)$
0, 1, $\vdots$, $(n-1-k)$	All mesh points	Mesh points on horizontal boundaries, excluding *reentrant* corners
$(n-k)$, $\vdots$, $(n-1)$	Mesh points on vertical boundaries, excluding *reentrant* corners	Four corners of S

Proof. It is well known [5] that the dimension of $H^n(\mathscr{R}, \pi)$ is $n^2 M$, where M is the total number of mesh points of π. With each mesh point $(x_i, y_j) \in \pi$ we can associate the n^2 basis elements $\phi_{ir}(x)\, \psi_{js}(y) : 0 \leqslant r,\ s \leqslant n - 1$, $(x, y) \in \mathscr{R}$, where $\phi_{ir}(x)$ and $\psi_{js}(y)$ are defined as in (2) with $k = 0$. Thus, if $p(x, y) \in H^n(\mathscr{R}, \pi)$, then

$$p(x, y) = \sum_i \sum_j \sum_{r=0}^{n-1} \sum_{s=0}^{n-1} p_{ij}^{(r,s)} \phi_{ir}(x)\, \psi_{js}(y), \tag{7}$$

where the summation on i and j is over all values such that $(x_i, y_j) \in \pi$.

Remark. Since $S_k{}^n(\mathscr{R}, \pi) \subseteq H^n(\mathscr{R}, \pi)$ for $0 \leqslant k \leqslant n-1$, each function in $S_k{}^n(\mathscr{R}, \pi)$ has a unique representation in terms of the above basis.

Fix k, $0 \leqslant k \leqslant n-1$. Define $B(f, k) = \{p(x, y) \in H^n(\mathscr{R}, \pi) : p_{ij}^{(r,s)} = f_{ij}^{(r,s)}$ for the values specified in Table II$\}$. We shall show that $B(f, k) \cap S_k{}^n(\mathscr{R}, \pi) = \{p_k\}$, i.e., there exists a unique element of $S_k{}^n(\mathscr{R}, \pi)$ which interpolates to the prescribed set of values given in Table II. If $k = 0$, i.e., $S_k{}^n(\mathscr{R}, \pi) = H^n(\mathscr{R}, \pi)$, then all n^2M parameters are specified in Table 2 and it follows that $B(f, k)$ consists of a single element, the smooth Hermite interpolant of f, [5]. For $0 < k \leqslant n-1$, $B(f, k)$ is a class of piecewise polynomials in which parameters not specified in Table II can be chosen arbitrarily. We now show that there is a unique set of values for these "free" parameters which yield a function of class $C^{(n-1+k, n-1+k)}(\mathscr{R})$, i.e., $B(f, k) \cap S_k{}^n(\mathscr{R}, \pi) = \{p_k\}$. We accomplish this by constructing *univariate* piecewise polynomials along mesh lines to which we apply Lemma 1.

Let I_j denote the horizontal mesh line $y = y_j$. The set of values $V_0(i, j) \equiv \{p_{ij}^{(r,s)} : 0 \leqslant r, s \leqslant n-1-k\}$ is given at each mesh point and the set of values $V_1(i, j) = \{p_{ij}^{(r,s)} : n-k \leqslant r \leqslant n-1, 0 \leqslant s \leqslant n-1-k\}$ is given at the end points of I_j. If $p \in B(f, k)$ and $p^{(0,s)}(x, y_j) \in C^{n-1+k}[I_j]$, $0 \leqslant s \leqslant n-1-k$, then the above sets of values $V_0(i, j)$ and $V_1(i, j)$ uniquely determine (see Lemma 1) the heretofore "free" values or parameters in $V_1(i, j)$ at each interior mesh point. Defining

$$B_1(f, k) \equiv B(f, k) \bigcap_j C^{(n-1+k)}[I_j],$$

$p \in B_1(f, k)$ implies that the sets of values $V_0(i, j)$ and $V_1(i, j)$ are determined at each mesh point.

Similarly, define

$$B_2(f, k) \equiv B_1(f, k) \bigcap_i C^{(n-1+k)}[I_i],$$

where I_i denotes the vertical mesh line $x = x_i$. Then applying Lemma 1 to $p^{(r,0)}(x_i, y)$, $0 \leqslant r \leqslant n-1-k$, the set of heretofore "free" parameters $V_2(i, j) \equiv \{p_{ij}^{(r,s)} : 0 \leqslant r \leqslant n-1-k,\ \ n-k \leqslant s \leqslant n-1\}$ is uniquely determined at each mesh point. Thus $p \in B_2(f, k)$ implies that the sets of values $V_0(i, j)$, $V_1(i, j)$ and $V_2(i, j)$ have been determined at each mesh point.

We have shown that $B_2(f, k)$ consists of all functions in $B(f, k)$ with only the set of values $V_3(i, j) \equiv \{p_{ij}^{(r,s)} : n-k \leqslant r, s \leqslant n-1\}$ "free" at each mesh point except the four amenable corner points. To show that the only element in $B(f, k) \cap S_k{}^n(\mathscr{R}, \pi)$, or $B(f, k) \cap C^{(n-1+k, n-1+k)}[\mathscr{R}]$ is p_k, it remains to prove that these parameters in $V_3(i, j)$ are also uniquely determined by the condition

$$p \in B(f, k) \cap S_k{}^n(\mathscr{R}, \pi) = B_2(f, k) \cap C^{(n-1+k, n-1+k)}[\mathscr{R}].$$

Let Q_1, Q_2, Q_3, and Q_4 be the four given amenable corner points and assume (without loss of generality) that Q_1 and Q_2 determine the mesh line $y = y_j$. Then:

Step 1. At the points Q_1 and Q_2 of the mesh line $y = y_j$, the values in $V_4(i,j) \equiv \{p_{ij}^{(r,s)} : 0 \leqslant r, s \leqslant n - 1\}$ are known. The values in $V_2(i,j) \cup V_0(i,j)$ have been determined at each interior mesh point of $y = y_j$. By Theorem 2, for each s, $n - k \leqslant s \leqslant n - 1$, $p^{(0,s)}(x, y_j) \in C^{n-1+k}[I_j]$. By Lemma 1, $p^{(0,s)}(x, y_j)$ is uniquely determined, i.e., for each s, $n - k \leqslant s \leqslant n - 1$, the values $p_{ij}^{(r,s)} : n - 1 - k < r \leqslant n - 1$ are uniquely determined at each mesh point on $y = y_j$.

The above procedure can be repeated for each pair (Q_i, Q_j), $1 < i$, $j \leqslant 4$, of amenable corner points which lie on the same mesh line. In so doing the values in $V_3(i,j)$ are uniquely determined for each point $(x_i, y_j) \in S_1$, where S_1, S_2,... are as defined prior to Definition 1.

Step 2. If $P_{ij} \in S_2$, then P_{ij} is on a mesh line, say $x = x_i$, containing two points, say P_1 and P_2, of S_1. The values in $V_3(i,j)$ are then uniquely determined at P_{ij}, using Lemma 1 with α and β specified by the y-coordinates of P_1 and P_2.

Step 3. Next consider $(x_i, y_j) \in S_3$. This point lies on a mesh line passing through two points of S_2. Lemma 1 can be applied to obtain values in $V_3(i,j)$ at each $(x_i, y_j) \in S_3$.

Considering S_4,..., S_N in order, we can obtain the values in $V_3(i,j)$ at each mesh point of π.

We now have determined a unique set of values $V_4(i,j)$ for each $(x_i, y_j) \in \pi$. We cannot yet conclude, however, that the p in (7) belongs to $S_k{}^n(\mathscr{R}, \pi)$ since for $n - k - 1 < r$, $s \leqslant n - 1$ the continuity of $p^{(0,s)}$ and $p^{(r,0)}$ along *all* mesh lines was not assured in Steps 1 through 3 above.

Consider the following illustrative example. Let $\mathscr{R}$ be the rectangular polygon in Fig. 1. The number associated with each mesh point (x_i, y_j) in Fig. 1 denotes the step at which the values in the set $V_3(i,j)$ were determined. The question arises: Are the values in $V_3(i,j)$ at points P_1, P_2, P_3 (determined in Step 3) *consistent* in the sense that for each r, $n - 1 - k \leqslant r \leqslant n - 1$, $p^{(r,0)}(x_m, y) \in C^{n-1+k}[\overline{P_1P_3}]$?

That they are consistent can be seen by the following application of Theorem 1.

After Step 3 the values in $V_4(i,j)$ have been determined at each boundary mesh point of the rectangular subregion $Q_1W_2W_1Q_3$. The values in $V_0(i,j)$ were given at each mesh point (x_i, y_j) and the values in $V_1(i,j)[V_2(i,j)]$ were *given or uniquely determined* at each mesh point on the vertical [horizontal] boundaries of $Q_1W_2W_1Q_3$. Hence, by applying Theorem 1 (with $\alpha_1 = \alpha_2 = 0$,

$\beta_1 = m, \beta_2 = m'$), there is a unique set of values $V_4(i, j)$ for each mesh point $(x_i, y_j) \in (Q_1W_2W_1Q_3, \pi)$ and an associated polynomial function $p_k \in C^{(n-1+k,n-1+k)}(Q_1W_2W_1Q_3)$. From the proof of Theorem 1 and the construction of $p(x, y)$ it follows that $p_k^{(r,s)}(x_i, y_j) = p_{ij}^{(r,s)}$, $0 \leqslant r, s \leqslant n - 1$, for each point $(x_i, y_j) \in Q_1W_2W_1Q_3$.

We next consider the subrectangle $Q_3W_4P_1P_3$. The values in $V_0(i, j)$ were given at each mesh point (x_i, y_j) of $Q_3W_4P_1P_3$ and values in $V_1(i, j)[V_2(i, j)]$ were given or uniquely determined along each vertical [horizontal] boundary. We now apply Theorem 1 (with $\alpha_1 = \alpha_2 = 0$, β_1 specified by the x-coordinate of W_3, β_2 specified by the y-coordinate of Q_3) to the rectangle $Q_3W_4P_1P_3$. Again note that the values in $V_4(i, j)$ obtained at each mesh point (x_i, y_j) using Theorem 1 agree with the corresponding values $p_{ij}^{(r,s)}$ used in Eq. (7). Thus the values in $V_3(i, j)$ determined in Step 3 for P_1, P_2, P_3 are consistent.

Repeated application of the above argument for the consistency of the values in $V_3(i, j)$ for the points P_1, P_2, P_3 establishes the consistency of the values in $V_3(i, j)$ for all points $(x_i, y_j) \in \pi$.

For each rectangular subregion considered, by Theorem 2 the polynomial function $p_k \in C^{(n-1+k,n-1+k)}$. But for each such rectangle, the polynomial function p constructed using Eq. (7) coincides with p_k. Thus $p \in C^{(n-1+k,n-1+k)}[\mathscr{R}]$.

We have proven $p \in S_k^n(\mathscr{R}, \pi)$ for the region $\mathscr{R}$ in Fig. 1. Clearly, a general rectangular polygon $\mathscr{R}$ can be treated in a similar manner.

Let $u \in B(f, k) \cap S_k^n(\mathscr{R}, \pi)$. Then $u - p_k$ belongs to $S_k^n(\mathscr{R}, \pi)$ and all parameters specified in Table 2 are zero. The univariate piecewise polynomials along the mesh lines are all zero; hence $u - p_k \equiv 0$, and the solution is unique. This completes the proof of Theorem 3.

We now discuss several interesting consequences of Theorem 3. For example, the heretofore unresolved question of the dim $S_k^n(\mathscr{R}, \pi)$ is answered by simply counting the number of interpolating conditions in each scheme—specifically,

THEOREM 4. *Let M be the total number of mesh points, C be the number of corners, E be the number of reentrant corners and B be the number of boundary mesh points, then*

$$\dim S_k^n(\mathscr{R}, \pi) = M(n-k)^2 + (B - 2E + C)\, k(n-k) + 4k^2.$$

It is well-known that for a rectangle $\mathscr{R}$, $S_k^n(\mathscr{R}, \pi)$ is a tensor product of spaces of univariate splines. As a consequence of Theorems 3 and 4 it is easy to construct examples of *rectangular polygons* $\mathscr{R}$ such that $S_k^n(\mathscr{R}, \pi)$ is not even a *subspace* of a tensor product of spaces of univariate splines.

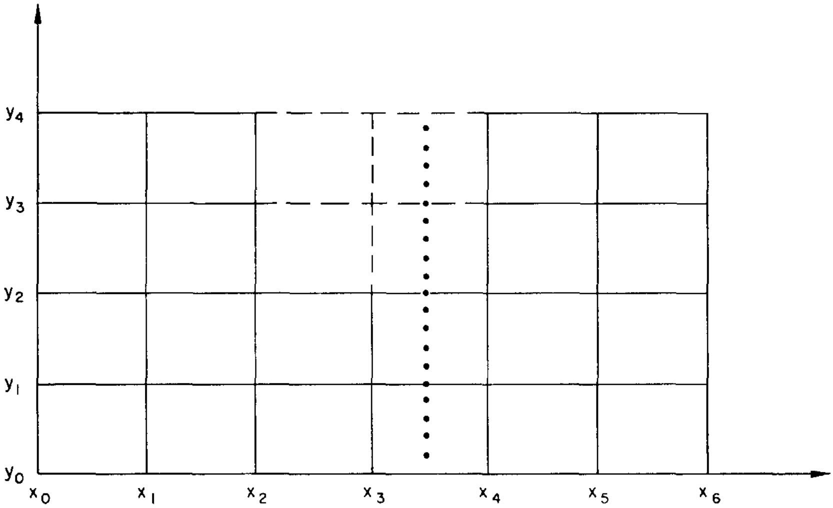


FIGURE 3

Example 1. Let $\mathscr{R}$ be the U-shaped region in Fig. 3; then $\dim S_1{}^2(\mathscr{R}, \pi) = 65$. But $\dim S_1{}^2(\bar{\mathscr{R}}, \bar{\pi}) = 63$ where $\bar{\mathscr{R}}$ is the smallest enclosing rectangle $[x_0, x_6] \times [y_0, y_4]$. That is, there exist bicubic splines on $\mathscr{R}$ which are *not* restrictions of bicubic splines on $\bar{\mathscr{R}}$.

Remark. This example is somewhat surprising, because by Whitney's Extension Theorem [14, 15], an $f \in C^4[\mathscr{R}]$ can be extended to an $F \in C^4[\bar{\mathscr{R}}]$ and the approximating spline $s \in S_1{}^2(\bar{\mathscr{R}}, \bar{\pi})$ to F can be restricted to an element in $S_1{}^2(\mathscr{R}, \pi)$. Thus to find the best approximation (in some sense) to f over $S_1{}^2(\mathscr{R}, \pi)$ it does not suffice to consider only splines in $S_1{}^2(\mathscr{R}, \pi)$ which are restrictions of splines in $S_1{}^2(\bar{\mathscr{R}}, \bar{\pi})$.

Note that if an additional (dotted) mesh line is inserted between x_3 and x_4, then $\dim S_1{}^2(\mathscr{R}, \pi) = \dim S_1{}^2(\bar{\mathscr{R}}, \bar{\pi}) = 70$ and each element of $S_1{}^2(\mathscr{R}, \pi)$ is the restriction of an element of $S_1{}^2(\bar{\mathscr{R}}, \bar{\pi})$.

4. VARIATIONAL PROPERTIES

The construction of Section 3 is clearly more complicated and artificial than it is for rectangles. Therefore, one wonders whether there may not be some simpler and more intrinsic way of characterizing spline functions, perhaps by variational properties. This variational characterization is well known for bicubic splines over rectangles [1, Theorem 7.6.1] and is given by

THEOREM 5. *Let $(\mathscr{R}, \pi)$ be a partitioned rectangle and let Γ be the set of all $u \in C^{(2,2)}(\mathscr{R})$ satisfying*

(i) $u(x_i, y_j) = f_{ij}$ *at each mesh point of* π;

(ii) $u^{(1,0)}(x_i, y_j) = f_{ij}^{(1,0)}$ *at each mesh point on a vertical boundary of* $\mathscr{R}$;

(iii) $u^{(0,1)}(x_i, y_j) = f_{ij}^{(0,1)}$ *at each mesh point on a horizontal boundary of* $\mathscr{R}$;

(iv) $u^{(1,1)}(x_i, y_j) = f_{ij}^{(1,1)}$ *at each corner of* $\mathscr{R}$.

Then the bicubic spline interpolant to the values (i)–(iv) *minimizes*

$$J_{\mathscr{R}}[u] = \iint_{\mathscr{R}} [u^{(2,2)}]^2 \, dx \, dy \tag{8}$$

over Γ.[2]

A salient feature of this variational characterization is frequently neglected, namely, that for any set of values given in (i)–(iv) above there *exists* a unique bicubic spline u_f in $S_1^2(\mathscr{R}, \pi)$ which interpolates to that set of values, i.e., $S_1^2(\mathscr{R}, \pi) \cap \Gamma = \{u_f\}$. This existence theorem is due to de Boor [7, Theorem 2]. The generalization of Theorem 5 to rectangular polygons appears to be immediate, since a rectangular polygon can be viewed as a union of rectangles. Indeed, based upon the comments in [1, p. 255] we have

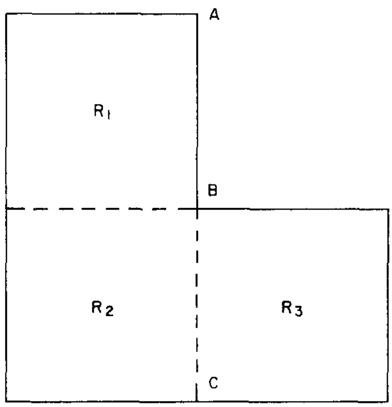
THEOREM 6. *Let $(\mathscr{R}, \pi)$ be a partitioned rectangular polygon such that $\mathscr{R} = \cup_{i=1}^{k} \mathscr{R}_i$ where each $\mathscr{R}_i$ is a rectangle. Let Γ be the class of all functions $u \in C^{(2,2)}(\mathscr{R})$ whose restrictions to $\mathscr{R}_i$ satisfy conditions* (i)–(iv) *for each rectangle $\mathscr{R}_i$. Then $\tilde{u}$ minimizes $J_{\mathscr{R}}[u]$ over Γ if and only if $\tilde{u}$ is the bicubic spline in $S_1^2(\mathscr{R}, \pi)$ which interpolates the given data* (i)–(iv) *in each $\mathscr{R}_i$.*

Remark. In a private communication with the authors of [1], it was revealed that in their monograph they were concerned mainly with the class $\Gamma' = C^{(1,1)}(\mathscr{R}) \cap C^{(2,2)}(\mathscr{R}_1) \cap \cdots \cap C^{(2,2)}(\mathscr{R}_k)$ rather than the class $\Gamma \subseteq C^{(2,2)}(\mathscr{R})$. Although such "piecewise splines" may prove to be more useful in numerical applications, it is our contention that "*bicubic splines*" should be in $C^{(2,2)}(\mathscr{R})$.

If Γ is replaced by Γ', or if Γ contains a bicubic spline, then Theorem 6 is an immediate consequence of the corresponding result for rectangles (Theorem 5). To complete the proof of Theorem 6 as stated, we show (Theorem 7) that Γ contains a bicubic spline if and only if the minimum of $J_{\mathscr{R}}[u]$ over Γ exists. First, however, we show that in general there exists *no* bicubic spline (in $C^2(\mathscr{R})$) whose restriction to each $\mathscr{R}_i$ interpolates to (i)–(iv); i.e., $S_1^2(\mathscr{R}, \pi) \cap \Gamma$ is empty. We illustrate with the following example.

[2] The authors were informed that Prof. Lois Mansfield was the first to notice that the uniqueness claimed in [1, p. 243] is false. Note that $J_R[u] = J_R[u + v]$ for any function v interpolating to zero data in (i)–(iv) and satisfying $v^{(2,2)} = 0$.

Example 2. Consider the L-shaped region R in Fig. 4. In [1, p. 254] it is proposed to fix the normal derivative on the boundaries,



R = R$_1$ ∪R$_2$ ∪R$_3$

FIGURE 4

and cross-derivatives at the four corners of each rectangle R_i, ($i = 1, 2, 3$). Thus along the line $\overline{AC}$, the derivatives $\partial s/\partial x$ are fixed at each mesh point and the derivatives $\partial^2 s/\partial x\, \partial y$ are fixed at A, B, and C. Clearly, this over-constrains the univariate spline $\partial s/\partial x$ along AC, and violates the necessary condition given in [2, p. 187] that $s \in C^2[\mathrm{R}]$. It is also clear that the number of given values to which s and its derivatives must interpolate exceeds the number $\dim S_1{}^2(\mathrm{R}, \pi)$, established in Theorem 4.

The above considerations imply that the proposed "spline" over R given in [1, p. 254] is in $C^2[\mathrm{R}_i]$ ($i = 1, 2, 3$) and $C^1[\mathrm{R}]$ and not in $C^2[\mathrm{R}]$. Thus in general, for an L-shaped region, there does *not* exist a bicubic spline which satisfies the set of boundary conditions specified in [1, p. 255].

To complete the proof of Theorem 6 we next show that the nonexistence of a bicubic spline in Γ is equivalent to the nonexistence of a minimum of $J_{\mathscr{R}}[u]$ over Γ, which is the contrapositive of the following

THEOREM 7. *The minimum of $J_{\mathscr{R}}[u]$ over Γ exists if and only if Γ contains a bicubic spline.*

Proof. Clearly,

$$J_{\mathscr{R}}[u] = \sum_{i=1}^{k} J_{\mathscr{R}_i}[u], \tag{9}$$

and

$$\inf_{u\in\Gamma} J_{\mathscr{R}}[u] \geqslant \sum_{i=1}^{k} \min_{u_i\in\Gamma_i} J_{\mathscr{R}}\,[u_i] = \sum_{i=1}^{k} J_{\mathscr{R}_i}[s_i] \tag{10}$$

where Γ_i is the set defined for the rectangle $\mathscr{R}_i$ as in Theorem 5, and s_i is the spline interpolant in Γ_i.

If Γ contains a bicubic spline $\tilde{u}$, its restriction $\tilde{u}_i$ to $\mathscr{R}_i$ is in $S_1^2(\mathscr{R}_i, \pi) \cap \Gamma_i$, and Theorem 5 implies equality in Eq. (10). That is, $\min_{u \in \Gamma} J_{\mathscr{R}}[u]$ exists.

We prove the converse only for the L-shaped region in Fig. 4. The proof for general $\mathscr{R}$ follows in a similar manner.

If $\tilde{u}$ minimizes $J_{\mathscr{R}}[u]$ over Γ and equality holds in Eq. (10) then the restriction of $\tilde{u}$ to each R_i is, in fact, s_i. Thus Γ contains the bicubic spline $\tilde{u}$.

If strict inequality holds in Eq. (10), then construct $u^* \in \Gamma$ as follows. Partition R_1 and R_2 as in Fig. 5.

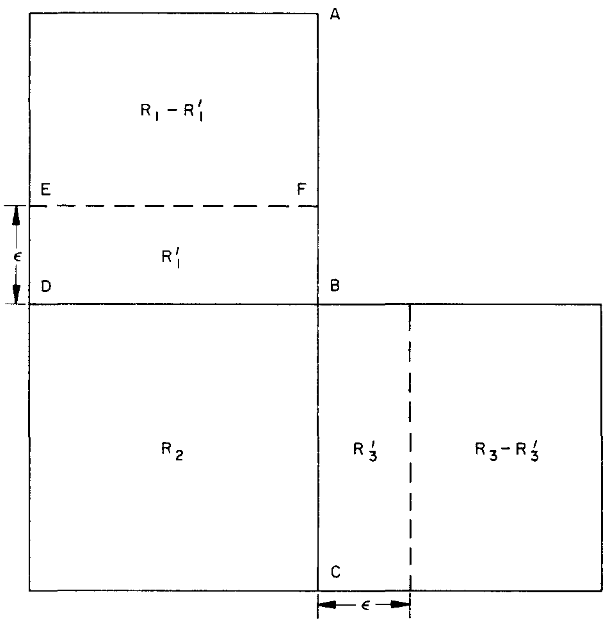


FIGURE 5

Define $u^* \equiv s_2$ in R_2, and $u^* \equiv s_i$ in $\mathrm{R}_i - \mathrm{R}_i'$ $(i = 1, 3)$. The rectangles R_1' and R_3' are naturally partitioned by π into a union of mesh cells. In R_1' define u^* to be the unique piecewise biquintic Hermite polynomial which interpolates to s_1 data on $\overline{EF}$ and to s_2 data on $\overline{DB}$. Define u^* similarly in R_3'.

For ϵ sufficiently small, the u^* so constructed is in Γ and $J_{\mathscr{R}}[\tilde{u}] > J_{\mathscr{R}}[u^*]$ contradicting the existence of the minimum for $\tilde{u}$.

The equivalence in Theorem 7 allows us to interpret Theorem 6 as a variational characterization of bicubic splines over general rectangular polygons. Unlike Theorem 5, it suffers from the lack of an existence theorem corresponding to the given interpolation conditions. It would be highly desirable to replace the set of interpolation conditions in Theorem 6 with a set for which an existence theorem holds. The use of Theorem 3 as such an existence theorem is unsatisfactory since the interpolating spline does not minimize $J_{\mathscr{R}}[u]$ over the associated class Γ. This is illustrated by the following example.

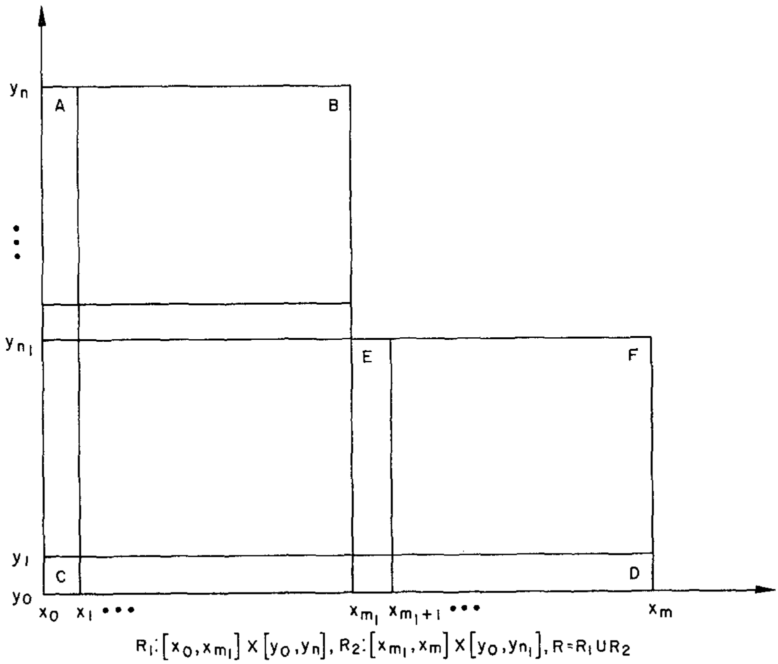


FIGURE 6

Example 3. For the L-shaped region in Fig. 6, let $m = 3, m_1 = 1, n_1 = 1$, $n = 2$ where the mesh π is uniform with mesh size h. Set all the interpolating values $f_{ij}^{(r,s)}$ in Theorem 3 to be zero except that $f_{12}^{(1,1)} = 1$ at the point B. (A, B, C, and D are the amenable corners.) For the bicubic spline p_1 determined in Theorem 3, one can compute that

$$\iint_{\mathrm{R}} (p_1^{(2,2)})^2 \, dx \, dy = 208/h^2$$

whereas for the piecewise *bi-quintic* basis function $g(x, y) = \phi_{11}(x) \, \psi_{21}(y)$, in $H^3(\mathrm{R}, \pi)$, which interpolates to $f_{12}^{(1,1)}$, we have

$$\iint_{\mathrm{R}} (g^{(2,2)})^2 \, dx \, dy = (32.49)/h^2.$$

5. An Analytically Well-Set Scheme

In Section 3, we provided algebraically well-set interpolation schemes for polynomial "splines" of any order and smoothness defined over general rectangular polygons. However, such interpolation schemes are not in general "analytically well-set". For example, a sequence of bicubic spline ($n = 2, k = 1$) interpolants to a smooth function need *not* converge as the mesh is successively refined. The difficulty involves the *extrapolation* of the cross-derivatives along $\overline{EF}$ (or $\overline{BE}$), (see [2, Appendix A] or [8, p. 15]).

Using the ideas of [10, p. 434] which were modified in [8], we now give a *convergent* interpolation scheme for bicubic spline interpolation over an L-shaped region. (We note that Birkhoff and de Boor [2, p. 187] doubted the existence of such a scheme.) This scheme differs from the scheme in Theorem 3 in that f_{xy} rather than f_y is interpolated along[3] the boundary segment $\overline{EF}$.

Theorem 8 *Let* (R, π) *be an L-shaped region (cf. Fig.* 6). *Let there be given* (i) *functional values* s_{ij} *at each mesh point,* (ii) $s_{ij}^{(1,0)}$ *along* $\overline{AC}$, $\overline{BE}$, $\overline{FD}$ *and at the corners A, B, C, D, and F,* (iii) $s_{ij}^{(0,1)}$ *along* $\overline{AB}$ *and* $\overline{CD}$, *and at the corners A, B, C, and D, and* (iv) $s_{ij}^{(1,1)}$ *along* $\overline{EF}$ *and at the corners A, C, D, and F. Then these values uniquely determine a bicubic spline interpolant* $s \in S_1{}^2(\mathrm{R}, \pi)$.

Proof. Theorem 2 implies $s \in C^{(2,2)}[\mathrm{R}]$, and the four values $s_{ij}^{(k,l)} : 0 \leqslant k, l \leqslant 1$ can be computed for each $(x_i, y_j) \in \pi$ by constructing univariate splines as outlined in Table III.

TABLE III

Step	Univariate spline(s) constructed	Values computed/given
1	$s(x, y_j),\ 0 \leqslant j \leqslant n$	$s_{ij}^{(1,0)}$ for all $(x_i, y_j) \in \pi$
2	$s(x_i, y),\ 0 \leqslant i \leqslant m_1$	$s_{ij}^{(0,1)}$ for all $(x_i, y_j) \in \pi \cap \mathrm{R}_1$
3	$s^{(0,1)}(x, y_j),\ j = 0, n$	$s_{ij}^{(1,1)}$ for $j = 0,\ 0 \leqslant i \leqslant m$ $j = n,\ 0 \leqslant i \leqslant m_1$
4	$s^{(0,1)}(x_i, y),\ 1 \leqslant i \leqslant m$	$s_{ij}^{(1,1)}$ for all $(x_i, y_j) \in \pi$
5	$s^{(0,1)}(x, y_{n_1}),\ x_{m_1-1} \leqslant x \leqslant x_m$	$s_{in}^{(0,1)}$ for $m_1 + 1 \leqslant i \leqslant m$
6	$s(x_i, y),\ m_1 + 1 \leqslant i \leqslant m$	$s_{ij}^{(0,1)}$ for all $(x_i, y_j) \in \pi \cap \mathrm{R}_2$

[Note that step 5 is the crucial step. For $y = y_{n_1}$, the functional values $s_{in_1}^{(0,1)}$ ($i = m_1 - 1, m_1$) and derivatives $s_{in_1}^{(1,1)}$ ($m_1 - 1 \leqslant i \leqslant m$) are given

[3] The term "along" excludes the end points.

and one can solve (stably) by [8, Lemma 2] for the values $s_{in_1}^{(0,1)}$ $(m_1 + 1 \leqslant i \leqslant m)$.]

The consistency of the above values follows as in the proof of Theorem 3. The proof of Theorem 8 is now complete.

The order of approximation of this interpolating bicubic spline was established in [8, Theorem 6]. For completeness we state this result below. [For a rectangular mesh π, let $\bar{h} = \max(x_i - x_{i-1})$, $\underline{h} = \min(x_i - x_{i-1})$, $\bar{h}' = \max(y_j - y_{j-1})$, $\underline{h}' = \min(y_j - y_{j-1})$, $h = \max(\bar{h}, \bar{h}')$, $\beta = \bar{h}/\underline{h}$ and $\beta' = \bar{h}'/\underline{h}'$.]

THEOREM 9. *Let* (R, π) *be a partitioned L-shaped region* (*Fig.* 6). *Let* $f \in C^4[\mathrm{R}]$ *and* $f^{(0,1)}(x, y_{n_1}) \in C^4[x_{n_1-1}, x_n]$. *Construct* $s \in S_1^2(\mathrm{R}, \pi)$ *as in Theorem* 8. *If* $(\bar{h}/\underline{h}')$ *and* $(\bar{h}'/\underline{h})$ *remain bounded as* $h \to 0$, *then, in* R_1,

$$\| s^{(k,l)} - f^{(k,l)} \|_\infty = O(h^{4-(k+l)}), \qquad 0 \leqslant k + l \leqslant 3 \tag{11}$$

and, in R_2,

$$\| s^{(k,l)} - f^{(k,l)} \|_\infty = O(h^{3-(k+l)}), \qquad 0 \leqslant k + l \leqslant 1. \tag{12}$$

If, in addition, $\beta = \beta' = 1$, $f \in C^5[\mathrm{R}]$ *and* $f^{(0,1)}(x, y_{n_1}) \in C^5[x_{n_1-1}, x_n]$, *then* (11) *holds throughout* R.

Remark. For a mesh which is uniform in each direction the bicubic spline s of Theorem 8 is a fourth-order approximation to f throughout the L-shaped region R.

Theorem 9 answers affirmatively a query of Birkhoff and de Boor [2, p. 187] concerning the existence of a convergent interpolation scheme for splines in L-shaped regions. However, the interpolation scheme of Theorem 9 is probably not optimal for a nonuniform partitioning of an L-shaped region. We suspect that one can construct, for such partitionings, interpolation schemes which are fourth-order approximations to smooth functions.

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